Structural Analysis III

Basis for the Analysis of Indeterminate Structures

2009/10

Dr. Colin Caprani

Contents

1. Introduction

1.1 Background

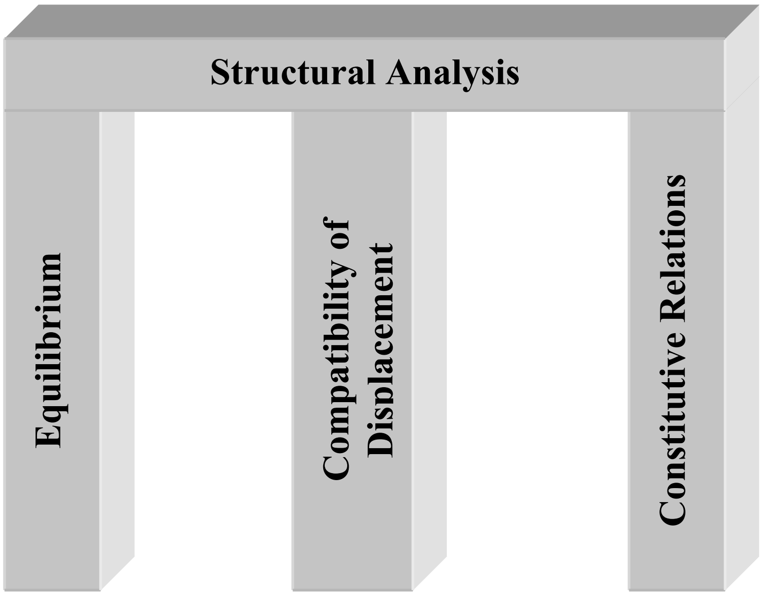
In the case of 2-dimensional structures there are three equations of statics:

$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum M = 0$$

Thus only three unknowns (reactions etc.) can be solved for using these equations alone. Structures that cannot be solved through the equations of static equilibrium alone are known as statically indeterminate structures. These, then, are structures that have more than 3 unknowns to be solved for. Therefore, in order to solve statically indeterminate structures we must identify other knowns about the structure.

1.2 Basis of Structural Analysis

The set of all knowns about structures form the basis for all structural analysis methods. Even if not immediately obvious, every structural analysis solution makes use of one or more of the three 'pillars' of structural analysis:



Equilibrium

Simply the application of the Laws of Statics – you have been using this pillar all along.

Compatibility of Displacement

This reflects knowledge of the connectivity between parts of a structure – as explained in this handout.

Constitutive Relations

The relationship between stress (i.e. forces moments etc) and strain (i.e. deflections, rotations) for the material in the structure being analysed. The *Principle of Superposition* (studied here) is an application of Constitutive Relations.

2. Small Displacements

2.1 Introduction

In structural analysis we will often make the assumption that displacements are *small*. This allows us to use approximations for displacements that greatly simplify analysis.

What do we mean by small displacements?

We take small displacements to be such that the arc and chord length are approximately equal. This will be explained further on.

Is it realistic?

Yes – most definitely. Real structures deflect very small amounts. For example, sways are usually limited to storey height over 500. Thus the arc or chord length is of the order 1/500th of the radius (or length of the member which is the storey height). As will be seen further on, such a small rotation allows the use of the approximation of small displacement.

Lastly, but importantly, in the analysis of flexural members, we ignore any changes in lengths of members due to axial loads. That is:

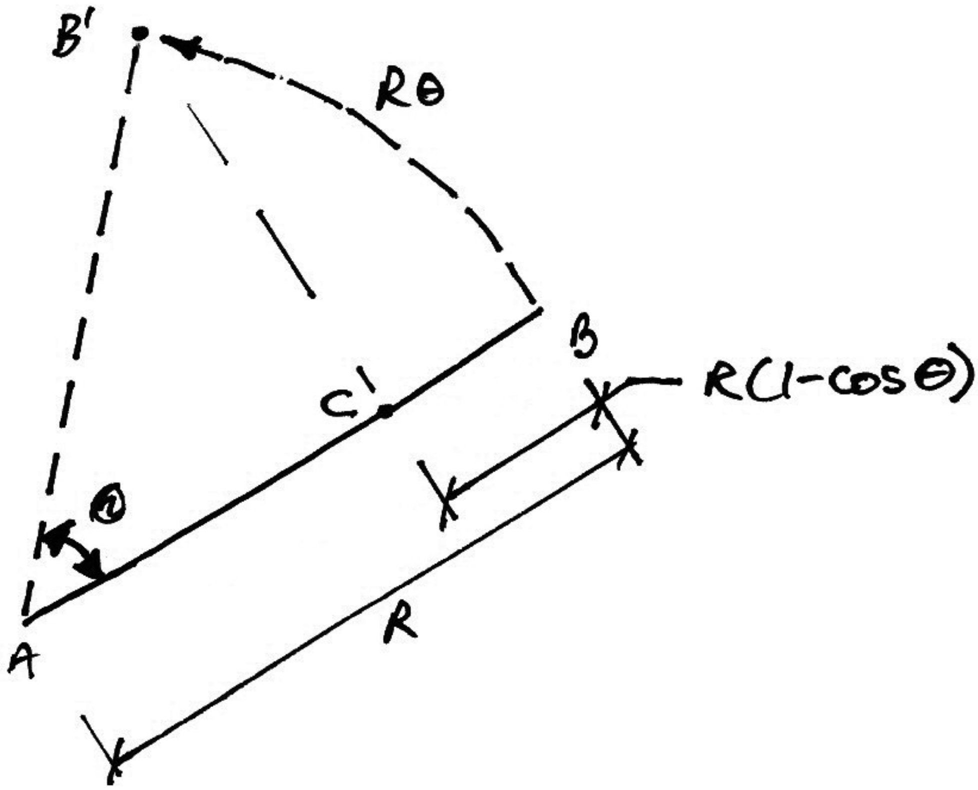
We neglect axial deformations – **members do not change length**.

This is because such members have large areas (as required for bending resistance) and so have negligible elastic shortening.

2.2 Derivation

Remember – all angles are in radians.

Consider a member *AB*, of length *R*, that rotates about *A*, an amount θ, to a new position *B*' as shown:



The total distance travelled by the point *B* is the length of the arc *BB*', which is $R\theta$.

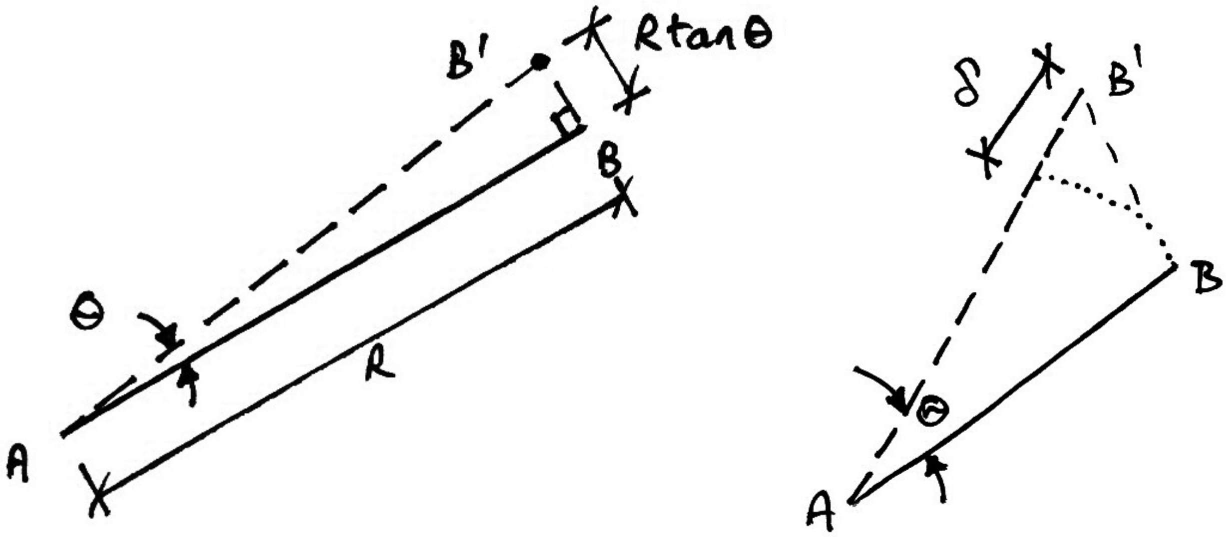
There is also the 'perpendicular distance' travelled by *B*: *CB*'. Obviously:

$$|CB'| < |BB'|$$
$$\text{Chord Length} < \text{Arc Length}$$
$$R\tan\theta < R\theta$$

There is also a movement of *B* along the line *AB*: *BC*, which has a length of:

$$R(1-\cos\theta)$$

Now if we consider a 'small' displacement of point *B*:



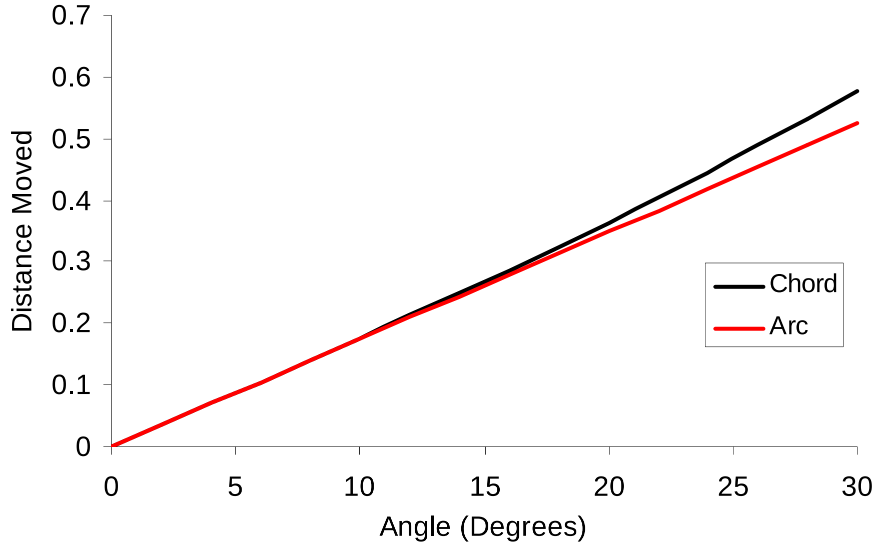
We can see now that the arc and chord lengths must be almost equal and so we use the approximation:

$$|BB'| = R\tan\theta \approx R\theta$$

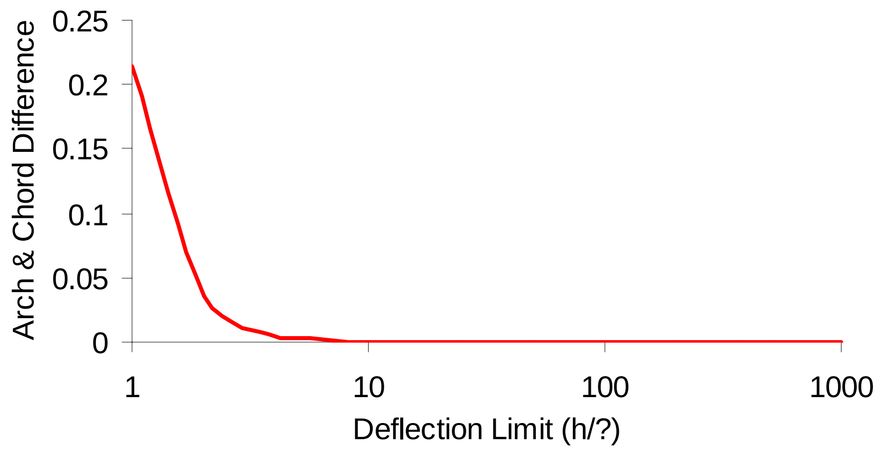
This is the approximation inherent in a lot of basic structural analysis. There are several things to note:

- It relies on the assumption that $\theta \approx \tan\theta$ for small angles;
- There is virtually no movement along the line of the member, i.e. $R(1-\cos\theta) \approx 0$ and so we neglect the small notional increase in length $\delta = |AB'| - |AB|$ shown above.

A graph of the arc and chord lengths for some angles is:



For usual structural movements (as represented by deflection limits), the difference between the arc and chord length approximation is:

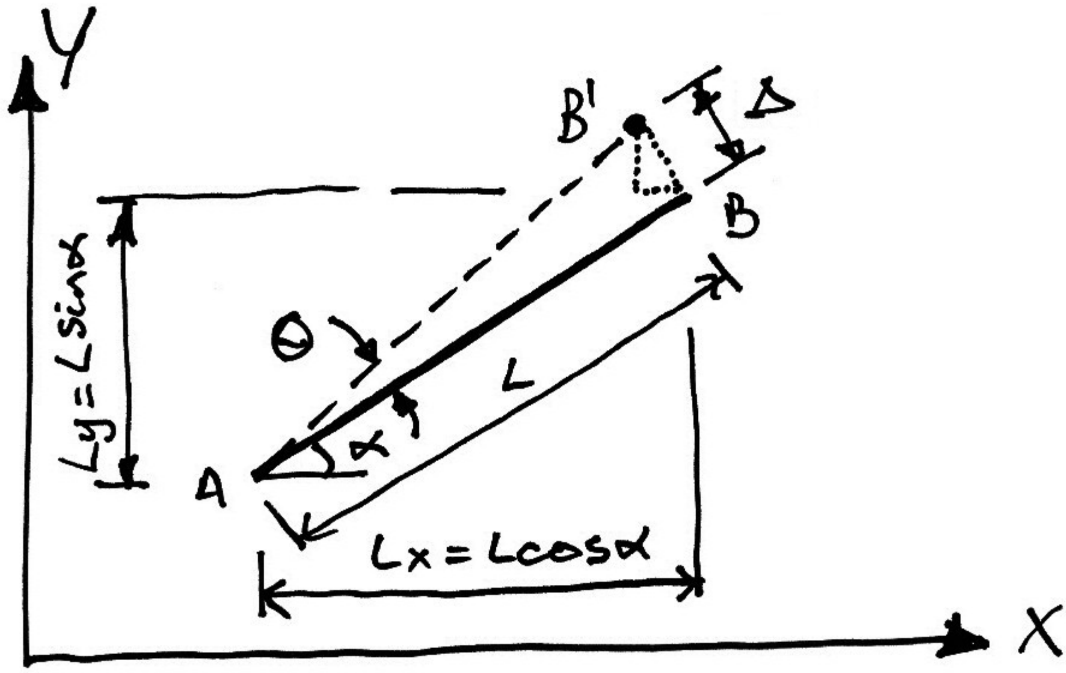


Since even the worst structural movement is of the order $h/200$ there is negligible difference between the arc and chord lengths and so the approximation of small angles holds.

2.3 Movement of Oblique Members

Development

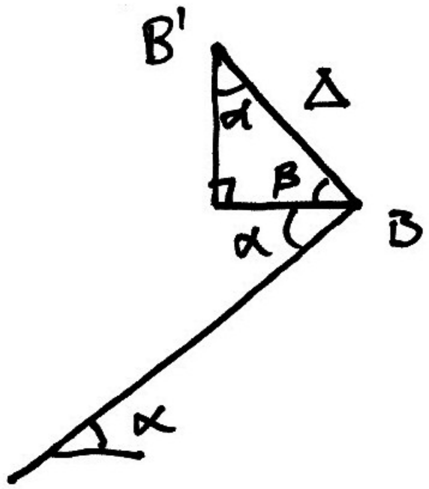
We want to examine the small rotation of an oblique member in the *x*-*y* axis system:



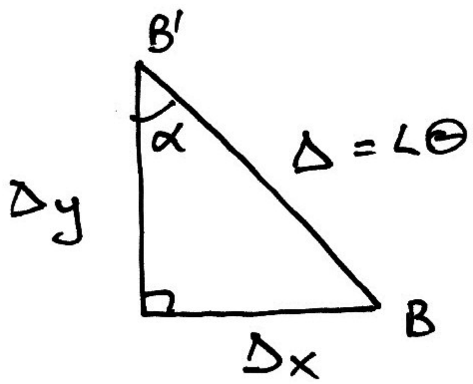
The member *AB*, which is at an angle α to the horizontal, has length *L* and undergoes a small rotation of angle θ about *A*. End *B* then moves to *B'* and by the theory of small displacements, this movement is:

$$\Delta = L\theta$$

We want to examine this movement and how it relates to the axis system. Therefore, we elaborate on the small triangle around *BB'* shown above, as follows:



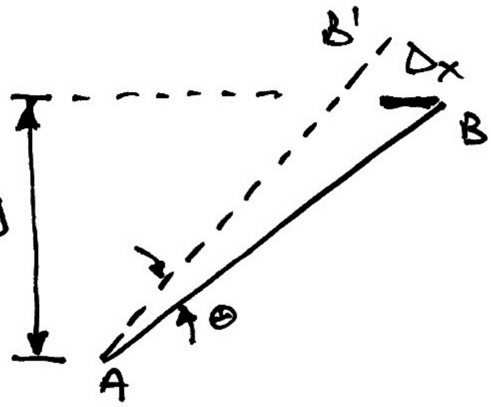
By using the rule: opposite angles are equal, we can identify which of the angles in the triangle is α and which is $\beta = 90° - \alpha$. With this knowledge we can now examine the components of the displacement Δ as follows:



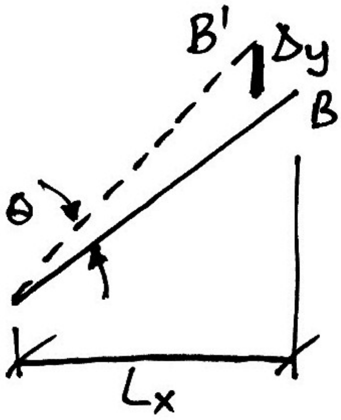
$$\begin{aligned}\Delta_Y &= \Delta \cos\alpha \\ &= L\theta\cos\alpha \\ &= L_X\theta\end{aligned}$$

$$\begin{aligned}\Delta_X &= \Delta \sin\alpha \\ &= L\theta\sin\alpha \\ &= L_Y\theta\end{aligned}$$

Therefore, the displacement of *B* along a direction (*x*- or *y*-axis) is given by the product of the rotation times the projection of the radius of movement onto an axis perpendicular to the direction of the required movement. This is best summed up by diagram:



$$\Delta_X = L_Y\theta$$

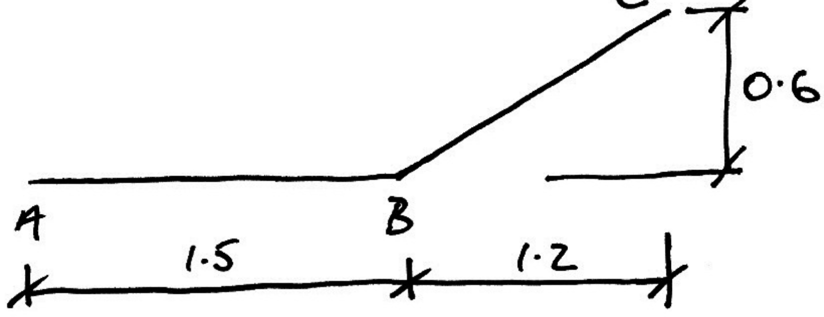


$$\Delta_Y = L_X\theta$$

Problem

For the following structure, show that a small rotation about *A* gives:

$$\Delta_{CX} = 0.6\theta; \quad \Delta_{CY} = 2.7\theta;$$
$$\Delta_{BX} = 0; \quad \Delta_{BY} = 1.5\theta$$



2.4 Instantaneous Centre of Rotation

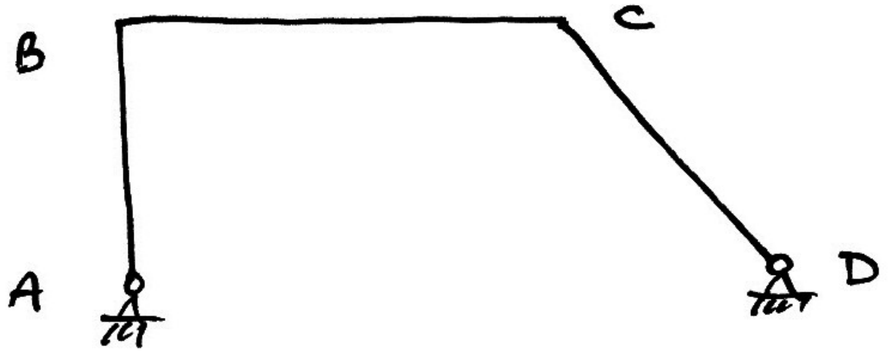
Definition

For assemblies of members (i.e. structures), individual members movements are not separable from that of the structure. A ‘global’ view of the movement of the structure can be achieved using the concept of the Instantaneous Centre of Rotation (ICR).

The Instantaneous Centre of Rotation is the point about which, for any given moment in time, the rotation of a body is occurring. It is therefore the only point that is not moving. In structures, each member can have its own ICR. However, movement of the structure is usually defined by an obvious ICR.

Development

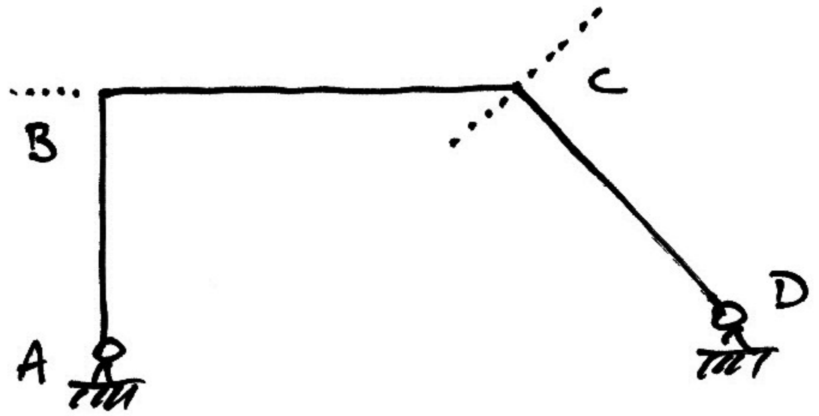
We will consider the deformation of the following structure:



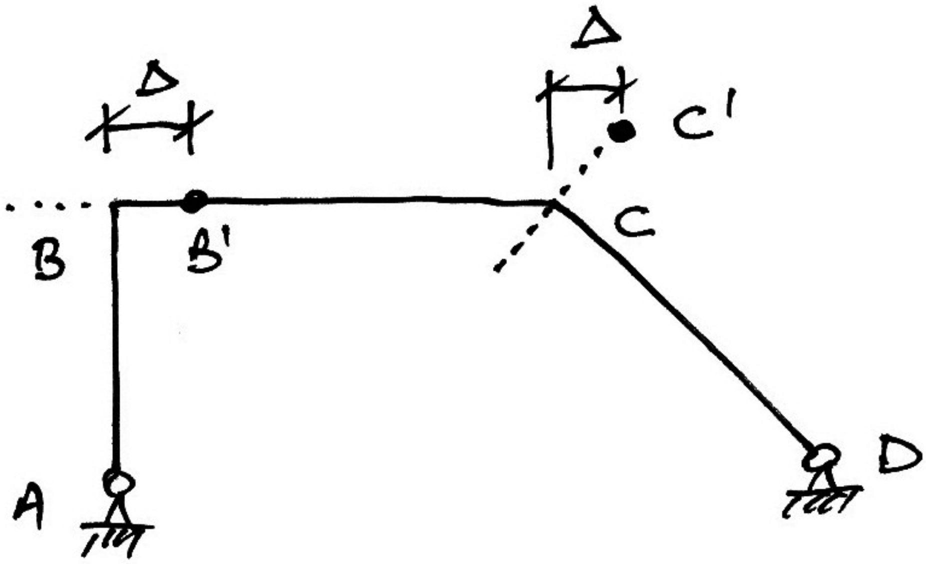
Firstly we must recognize that joints *A* and *D* are free to rotate but not move. Therefore the main movement of interest in this structure is that of joints *B* and *C*. Next we identify how these joints may move:

- Joint *B* can only move horizontally since member *AB* does not change length;
- Joint *C* can only move at an oblique angle, since member *CD* does not change length.

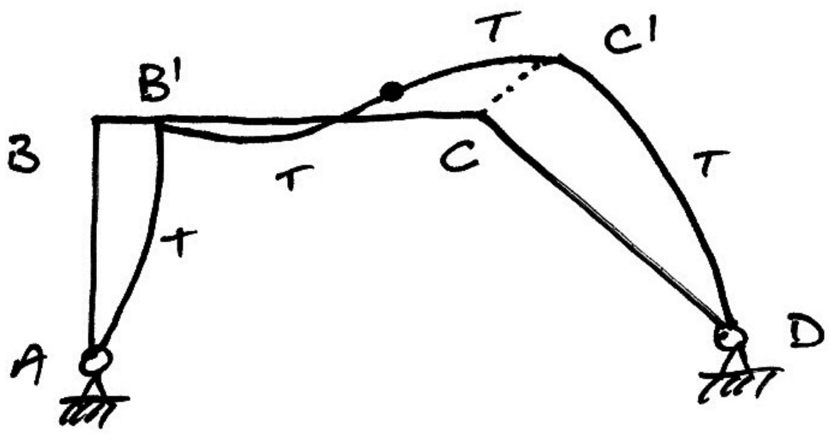
Thus we have the following paths along which the structure can move:



Next we take it that the loading is such that the structure moves to the right (we could just as easily have taken the left). Since member *BC* cannot change length either, the horizontal movements at joints *B* and *C* must be equal, call it Δ. Thus we have the deformed position of the joints *B* and *C*:

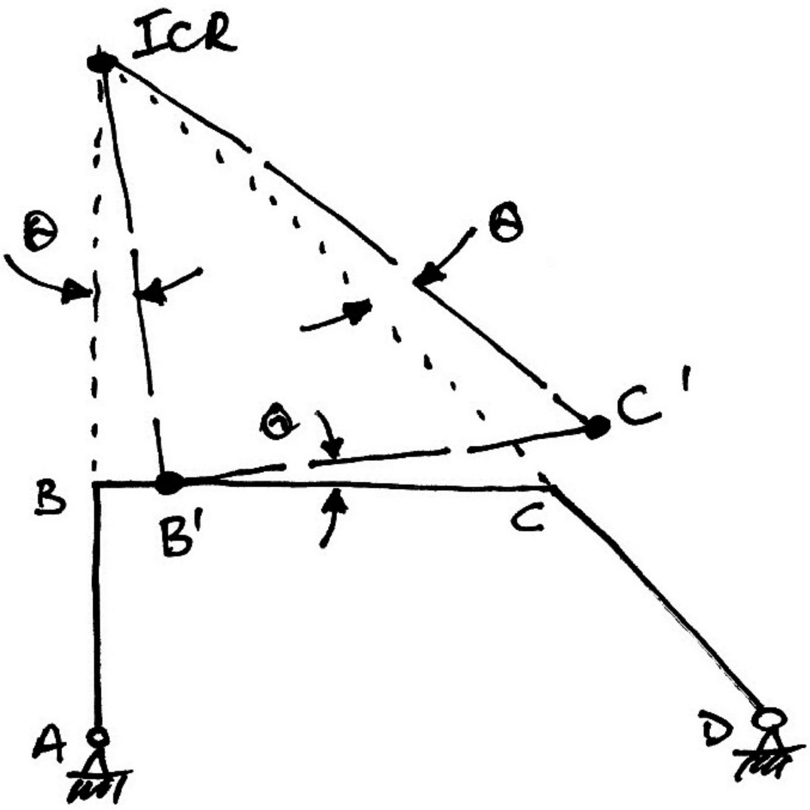


Now knowing these positions, we can draw the possible deflected shape of the structure, by linking up each of the deformed joint positions:



Looking at this diagram it is readily apparent that member *AB* rotates about *A* (its ICR) and that member *CD* rotates about *D* (its ICR). However, as we have seen, it is the movements of joints *B* and *C* that define the global movement of the structure. Therefore we are interested in the point about which member *BC* rotates and it is this point that critically defines the global movements of the structure.

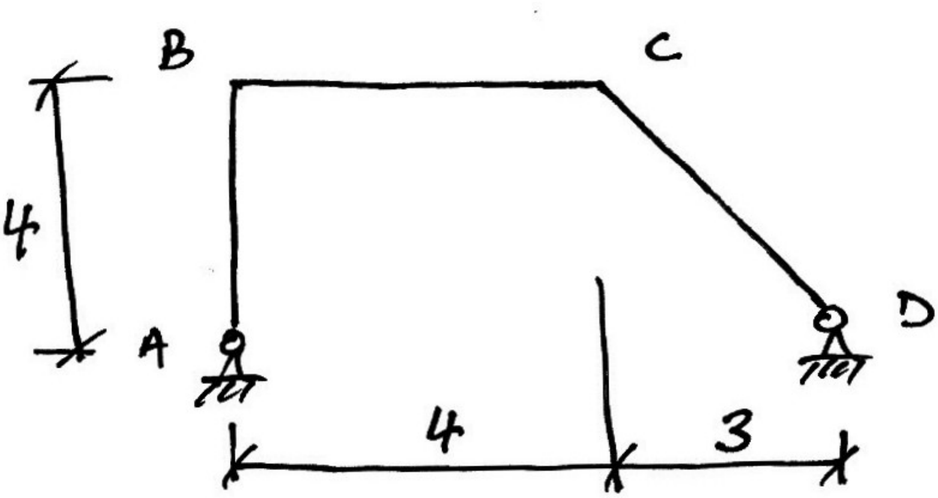
To find the ICR for member *BC* we note that since *B* moves perpendicular to member *AB*, the ICR for *BC* must lie along this line. Similarly, the line upon which the ICR must lie is found for joint *C* and member *CD*. Therefore, the ICR for member *BC* is found by producing the lines of the members *AB* and *CD* until they intersect:



From this figure, we can see that the movements of the structure are easily defined by the rotation of the lamina *ICR*-*B*-*C* about ICR by an angle θ.

Example

Find the relationship between the deflections of joints *B* and *C*.

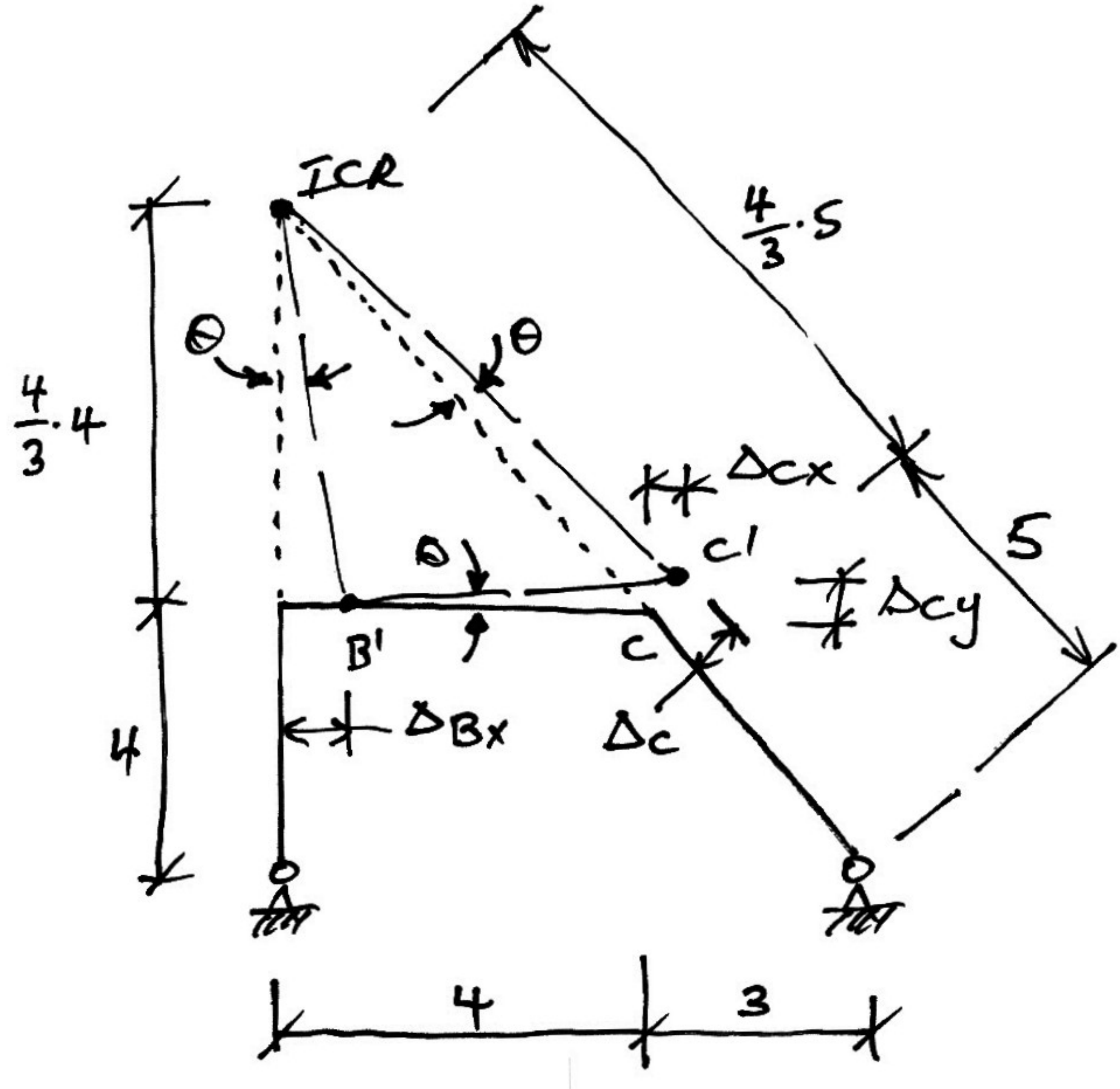


Our first step is to find the ICR by producing the lines of members *AB* and *CD*, as shown opposite.

Because of the angle of member *CD*, we can determine the dimensions of the lamina *ICR-B-C* as shown.

Next we give the lamina a small rotation about the ICR and identify the new positions of joints *B* and *C*.

We then work out the values of the displacements at joints *B* and *C* by considering the rule for small displacements, and the rotation of the lamina as shown.



$$\Delta_{BX} = \left(\frac{4}{3}\cdot 4\right)\theta = \frac{16}{3}\theta$$

$$\Delta_{C} = \left(\frac{4}{3}\cdot 5\right)\theta = \frac{20}{3}\theta$$

$$\Delta_{CX} = \Delta_{BX} = \frac{16}{3}\theta$$

$$\Delta_{CY} = 4\theta$$

3. Compatibility of Displacements

3.1 Description

When a structure is loaded it deforms under that load. Points that were connected to each other remain connected to each other, though the distance between them may have altered due to the deformation. All the points in a structure do this is such a way that the structure remains fitted together in its original configuration.

Compatibility of displacement is thus:

> *Displacements are said to be compatible when the deformed members of a loaded structure continue to fit together*.

Thus, compatibility means that:

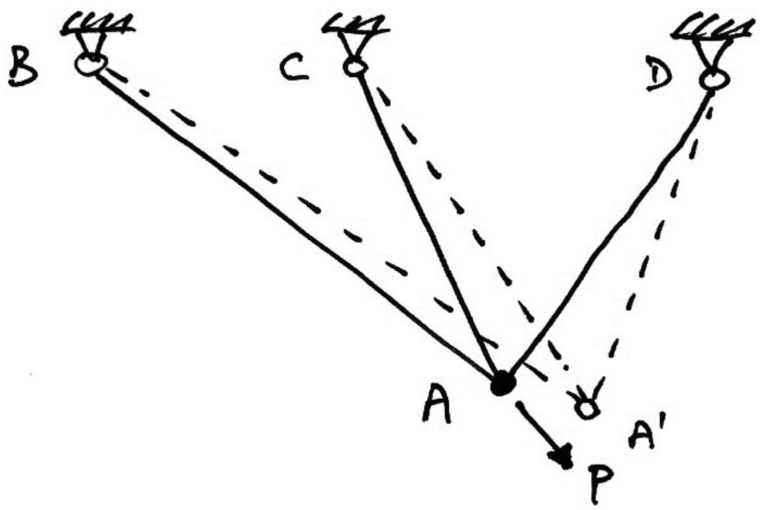
- Two initially separate points do not move to another common point;
- Holes do not appear as a structure deforms;
- Members initially connected together remain connected together.

This deceptively simple idea is very powerful when applied to indeterminate structures.

3.2 Examples

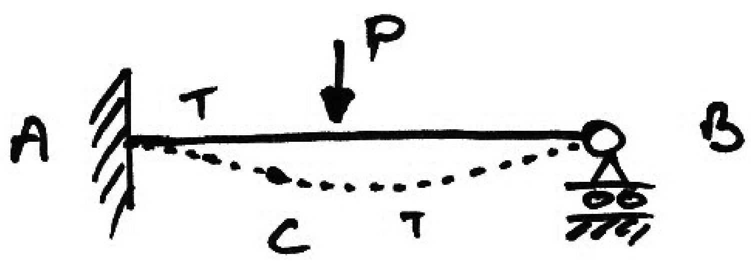
Truss

The following truss is indeterminate. Each of the members has a force in it and consequently undergoes elongation. However, by compatibility of displacements, the elongations must be such that the three members remain connected after loading, even though the truss deforms and Point *A* moves to Point *A'*. This is an extra piece of information (or 'known') and this helps us solve the structure.



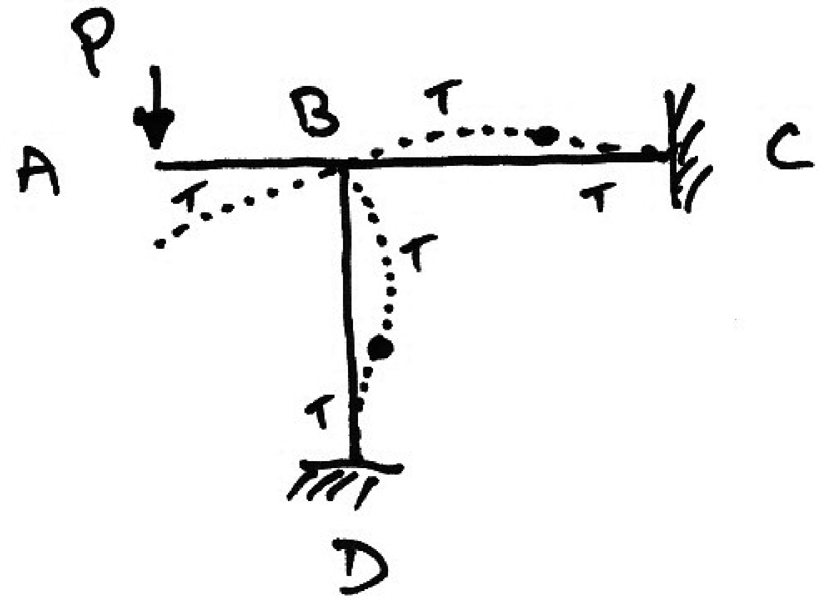
Beam

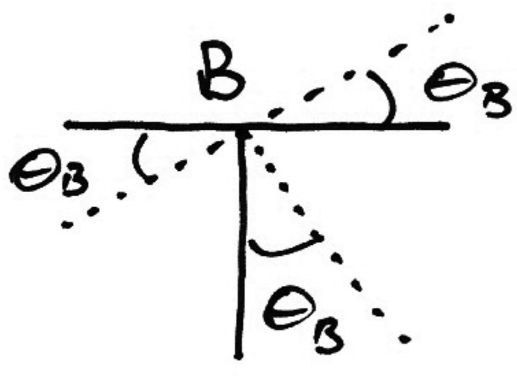
The following propped cantilever is an indeterminate structure. However, we know by compatibility of displacements that the deflection at point B is zero before and after loading, since it is a support.



Frame

The following frame has three members connected at joint *B*. The load at *A* causes joint *B* to rotate anti-clockwise. The ends of the other two members connected at *B* must also undergo an anti-clockwise rotation at *B* to maintain compatibility of displacement. Thus all members at *B* rotate the same amount, θ_B, as shown below.



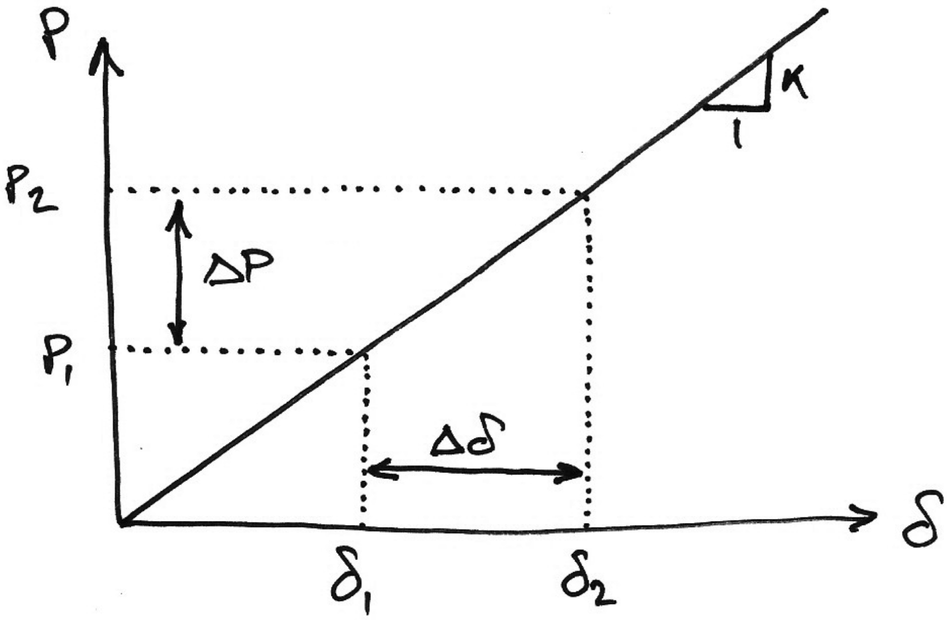


Joint *B*

4. Principle of Superposition

4.1 Development

For a linearly elastic structure, load, P, and deformation, δ, are related through stiffness, K, as shown:



For an initial load on the structure we have:

$$P_1 = K \cdot \delta_1$$

If we instead we had applied ΔP we would have gotten:

$$\Delta P = K \cdot \Delta\delta$$

Now instead of applying ΔP separately to P_1 we apply it after P_1 is already applied. The final forces and deflections are got by adding the equations:

$$\begin{aligned} P_1 + \Delta P &= K \cdot \delta_1 + K \cdot \Delta\delta \\ &= K\left(\delta_1 + \Delta\delta\right) \end{aligned}$$

But, since from the diagram, $P_2 = P_1 + \Delta P$ and $\delta_2 = \delta_1 + \Delta\delta$, we have:

$$P_2 = K \cdot \delta_2$$

which is a result we expected.

This result, though again deceptively ‘obvious’, tells us that:

- Deflection caused by a force can be added to the deflection caused by another force to get the deflection resulting from both forces being applied;
- The order of loading is not important (ΔP or P_1 could be first);
- Loads and their resulting load effects can be added or subtracted for a structure.

This is the **Principle of Superposition**:

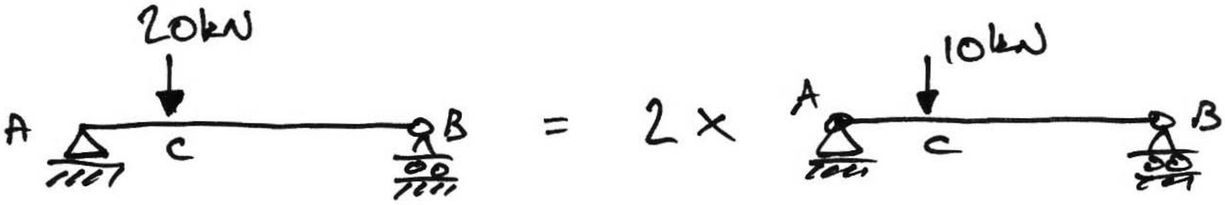
For a linearly elastic structure, the load effects caused by two or more loadings are the sum of the load effects caused by each loading separately.

Note that the principle is limited to:

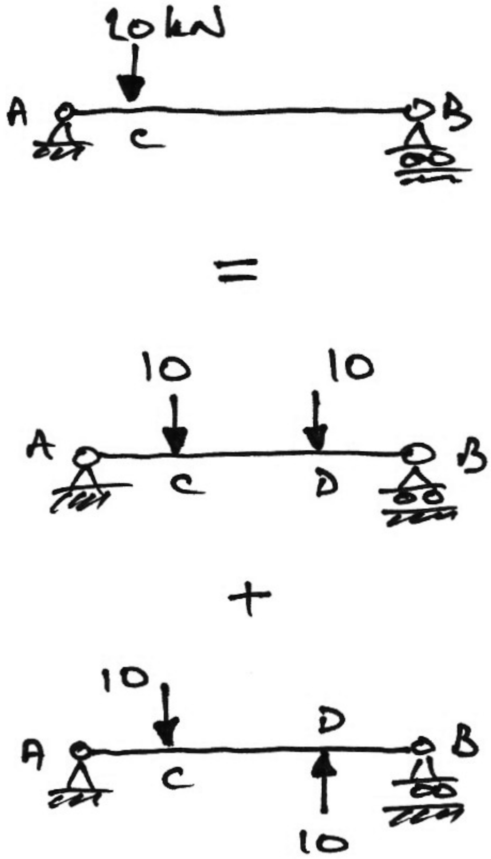
- Linear material behaviour only;
- Structures undergoing small deformations only (linear geometry).

4.2 Example

If we take a simply-supported beam, we can see that its solutions can be arrived at by multiplying the solution of another beam:



The above is quite obvious, but not so obvious is that we can also break the beam up as follows:



Thus the principle is very flexible and useful in solving structures.

5. Solving Indeterminate Structures

5.1 Introduction

There are two main approaches to the solution of indeterminate structures:

The Force Method

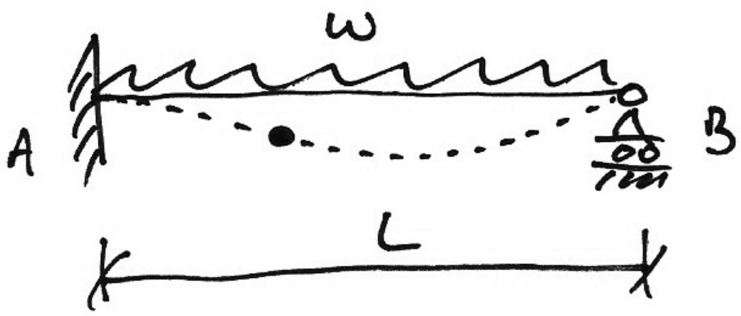
This was the first method of use for the analysis of indeterminate structures due to ease of interpretation, as we shall see. It is also called the compatibility method, method of consistent deformations, or flexibility method. Its approach is to find the redundant forces that satisfy compatibility of displacements and the force-displacement relationships for the structure's members. The fundamental ideas are easy to understand and we will use them to begin our study of indeterminate structures with the next few examples.

The Displacement Method

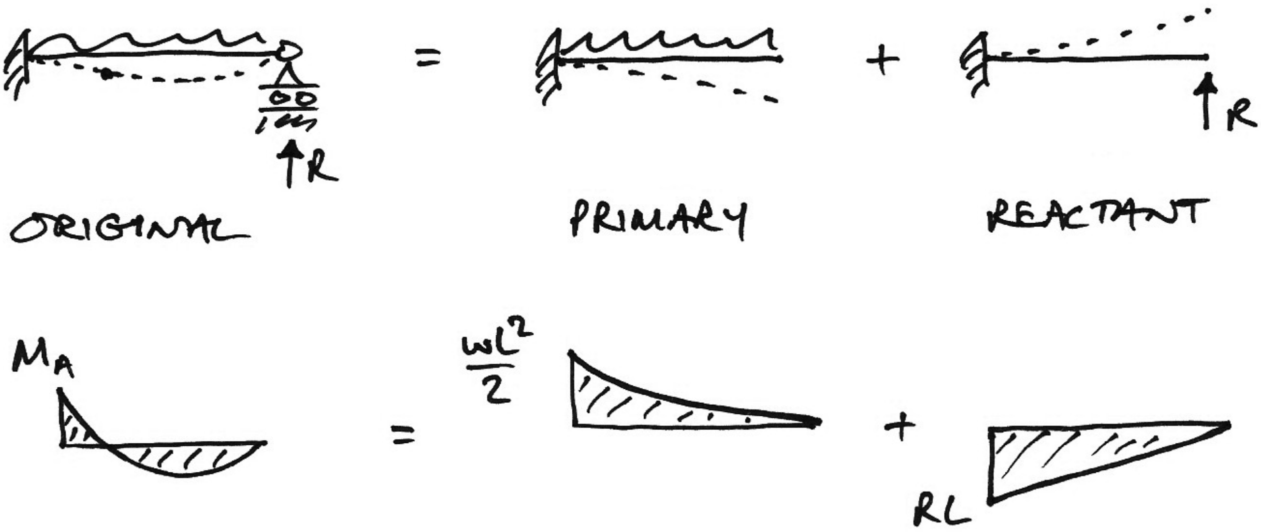
This method was developed later and is less intuitive than the force method. However, it has much greater flexibility and forms the basis for the finite-element method for example. Its approach is to first satisfy the force-displacement for the structure members and then to satisfy equilibrium for the whole structure. Thus its unknowns are the displacements of the structure. It is also called the stiffness method.

5.2 Example: Propped Cantilever

Consider the following propped cantilever subject to UDL:



Using superposition we can break it up as follows (i.e. we choose a redundant):



Next, we consider the deflections of the primary and reactant structures:

Now by compatibility of displacements for the original structure, we know that we need to have a final deflection of zero after adding the primary and reactant deflections at *B*:

$$\delta_B = \delta_B^P + \delta_B^R = 0$$

From tables of standard deflections, we have:

$$\delta_B^P = +\frac{wL^4}{8EI} \text{ and } \delta_B^R = -\frac{RL^3}{3EI}$$

In which downwards deflections are taken as positive. Thus we have:

$$\begin{aligned} \delta_B &= +\frac{wL^4}{8EI} - \frac{RL^3}{3EI} = 0 \\ \therefore R &= \frac{3wL}{8} \end{aligned}$$

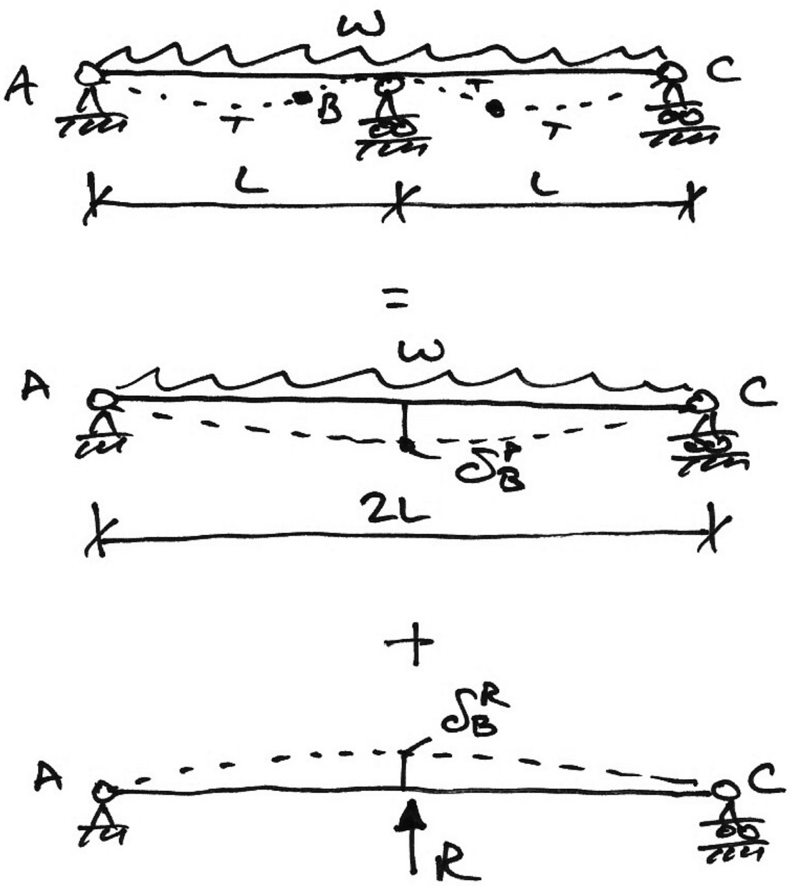
Knowing this, we can now solve for any other load effect. For example:

$$\begin{aligned} M_A &= \frac{wL^2}{2} - RL \\ &= \frac{wL^2}{2} - \frac{3wL}{8}L \\ &= \frac{4wL^2 - 3wL^2}{8} \\ &= \frac{wL^2}{8} \end{aligned}$$

Note that the $wL^2/8$ term arises without a simply-supported beam in sight!

5.3 Example: 2-Span Beam

Considering a 2-span beam, subject to UDL, which has equal spans, we break it up using the principle of superposition:



Once again we use compatibility of displacements for the original structure to write:

$$\delta_B = \delta_B^P + \delta_B^R = 0$$

Again, from tables of standard deflections, we have:

$$\delta_B^P = +\frac{5w(2L)^4}{384EI} = +\frac{80wL^4}{384EI}$$

And:

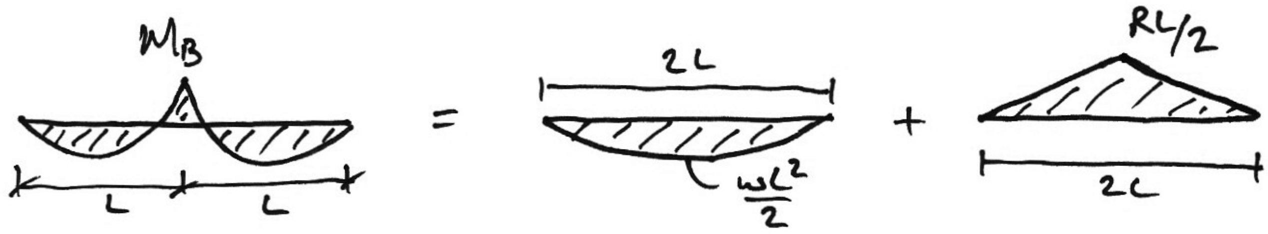
$$\delta_B^R = -\frac{R(2L)^3}{48EI} = -\frac{8RL^3}{48EI}$$

In which downwards deflections are taken as positive. Thus we have:

$$\begin{aligned}\delta_B &= +\frac{80wL^4}{384EI} - \frac{8RL^3}{48EI} = 0 \\ \frac{8R}{48} &= \frac{80wL}{384} \\ R &= \frac{10wL}{8}\end{aligned}$$

Note that this is conventionally not reduced to $5wL/4$ since the other reactions are both $3wL/8$. *Show this as an exercise.*

Further, the moment at *B* is by superposition:



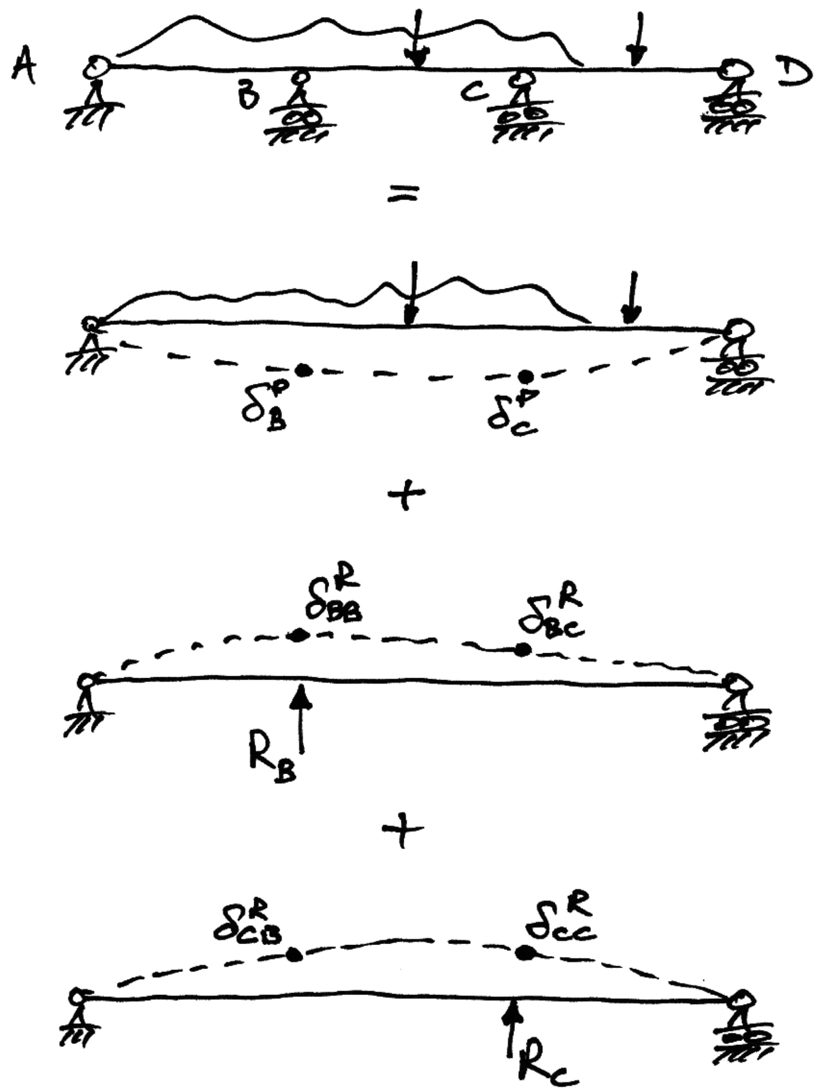
Hence:

$$\begin{aligned}M_B &= \frac{RL}{2} - \frac{wL^2}{2} = \frac{10wL}{8}\cdot\frac{L}{2} - \frac{wL^2}{2} = \frac{10wL^2 - 8wL^2}{16} \\ &= \frac{wL^2}{8}\end{aligned}$$

And again $wL^2/8$ arises!

5.4 Force Method: General Case

Let’s consider the following 2° indeterminate structure:



We have broken it up into its primary and redundant structures, and identified the various unknown forces and displacements.

We can express the redundant displacements in terms of the redundant forces as follows:

$$\delta_{BB}^{R} = R_B f_{BB} \qquad \delta_{BC}^{R} = R_B f_{BC}$$
$$\delta_{CC}^{R} = R_C f_{CC} \qquad \delta_{CB}^{R} = R_C f_{CB}$$

We did this in the last example too:

$$\delta_{B}^{R} = R_B \left(\frac{8L^3}{48EI} \right) = R_B f_{BB}$$

The coefficients of the redundant forces are termed flexibility coefficients. The subscripts indicate the location of the load and the location where the displacement is measured, respectively.

We now have two locations where compatibility of displacement is to be met:

$$\delta_{B}^{P} + \delta_{B}^{R} = 0$$
$$\delta_{C}^{P} + \delta_{C}^{R} = 0$$

As we can see from the superposition, the redundant displacements are:

$$\delta_{B}^{R} = \delta_{BB}^{R} + \delta_{CB}^{R}$$
$$\delta_{C}^{R} = \delta_{CC}^{R} + \delta_{BC}^{R}$$

And if we introduce the idea of flexibility coefficients:

$$\delta_{B}^{R} = R_B f_{BB} + R_C f_{CB}$$
$$\delta_{C}^{R} = R_C f_{CC} + R_B f_{BC} = R_B f_{BC} + R_C f_{CC}$$

Then the compatibility of displacement equations become:

$$\delta_B^P + R_B f_{BB} + R_C f_{CB} = 0$$
$$\delta_C^P + R_B f_{BC} + R_C f_{CC} = 0$$

Which we can express in matrix form:

$$\begin{Bmatrix} \delta_B^P \\ \delta_C^P \end{Bmatrix} + \begin{bmatrix} f_{BB} & f_{CB} \\ f_{BC} & f_{CC} \end{bmatrix} \begin{Bmatrix} R_B \\ R_C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

And in general we have:

$$\left\{ \boldsymbol{\delta}^P \right\} + \left[\mathbf{f} \right] \left\{ \mathbf{R} \right\} = \left\{ \mathbf{0} \right\}$$

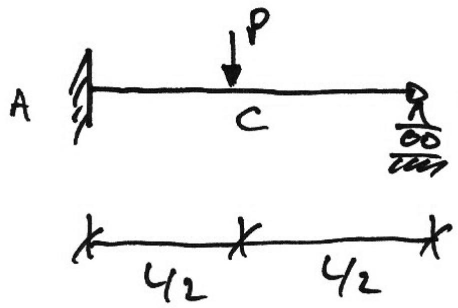
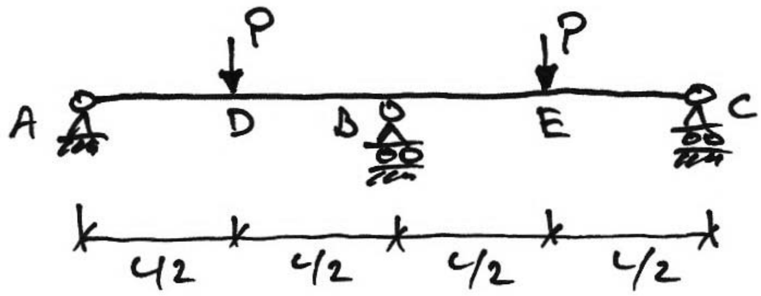
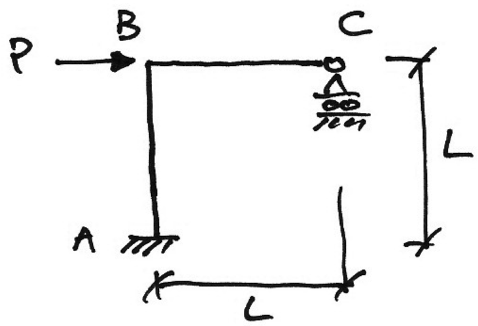
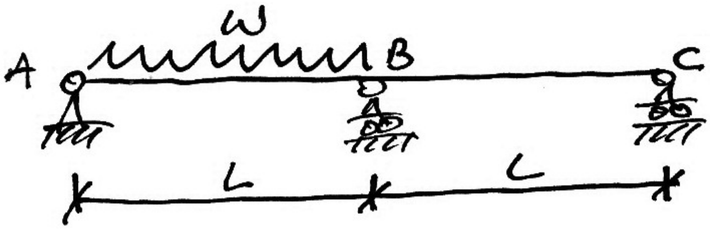
Since we know the primary structure displacements and the flexibility coefficients we can determine the redundants:

$$\left\{ \mathbf{R} \right\} = -\left[\mathbf{f} \right]^{-1} \left\{ \boldsymbol{\delta}^P \right\}$$

Thus we are able to solve a statically indeterminate structure of any degree.

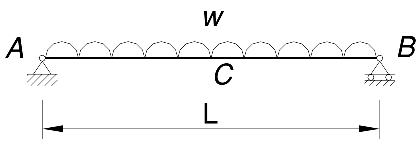
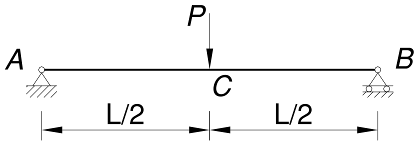
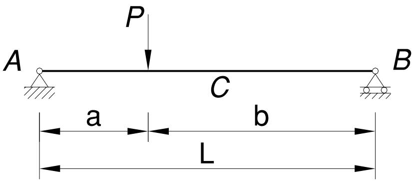
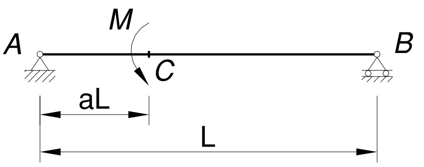
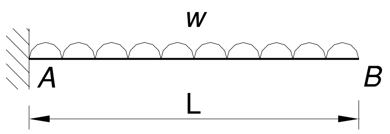
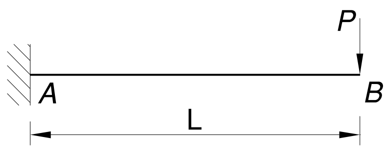
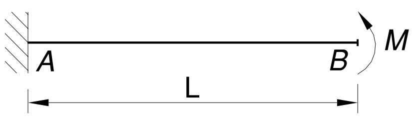
6. Problems

Use compatibility of displacement and the principle of superposition to solve the following structures. In each case draw the bending moment diagram and determine the reactions.

1.		$M_A = \frac{3PL}{16}$
2.		
3.		$V_C = 3P/8$
4.		$M_B = \frac{wL^2}{16}$

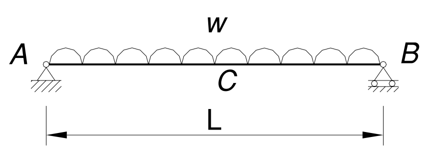
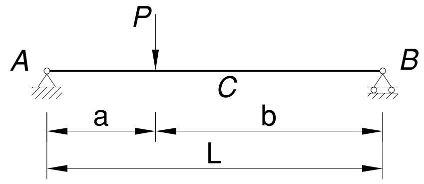
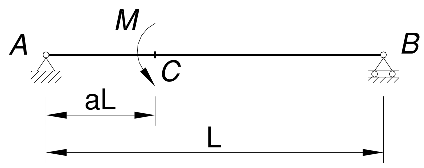
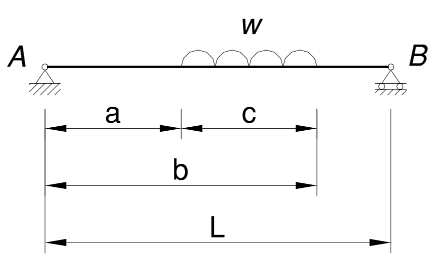
7. Displacements

Point Displacements

Configuration	Translations	Rotations
	$\delta_C = \frac{5wL^4}{384EI}$	$\theta_A = -\theta_B = \frac{wL^3}{24EI}$
	$\delta_C = \frac{PL^3}{48EI}$	$\theta_A = -\theta_B = \frac{PL^2}{16EI}$
	$\delta_C \cong \frac{PL^3}{48EI}\left[\frac{3a}{L} - 4\left(\frac{a}{L}\right)^3\right]$	$\theta_A = \frac{Pa(L-a)}{6LEI}(2L-a)$ $\theta_B = -\frac{Pa}{6LEI}\left(L^2 - a^2\right)$
	$\delta_C = \frac{ML^2}{3EI}a(1-a)(1-2a)$	$\theta_A = \frac{ML}{6EI}\left(3a^2 - 6a + 2\right)$ $\theta_B = \frac{ML}{6EI}\left(3a^2 - 1\right)$
	$\delta_B = \frac{wL^4}{8EI}$	$\theta_B = \frac{wL^3}{6EI}$
	$\delta_B = \frac{PL^3}{3EI}$	$\theta_B = \frac{PL^2}{2EI}$
	$\delta_B = \frac{ML^2}{2EI}$	$\theta_B = \frac{ML}{EI}$

General Equations

Coordinate x is zero at A and increases to the right. The right angled brackets evaluate to zero if the term inside is negative (called Macaulay brackets)

	$V_A = \frac{wL}{2}$ $EI\theta_A = -\frac{wL^3}{24}$ $EI\theta(x) = \frac{V_A}{2}x^2 - \frac{w}{6}x^3 + EI\theta_A$ $EI\delta(x) = \frac{V_A}{6}x^3 - \frac{w}{24}x^4 + EI\theta_A x$
	$V_A = \frac{Pb}{L}$ $EI\theta_A = -\frac{Pb}{6L}\left(L^2 - b^2\right)$ $EI\theta(x) = \frac{V_A}{2}x^2 - \frac{P}{2}\langle x-a\rangle^2 + EI\theta_A$ $EI\delta(x) = \frac{V_A}{6}x^3 - \frac{P}{6}\langle x-a\rangle^3 + EI\theta_A x$
	$V_A = \frac{M}{L}$ $EI\theta_A = -\frac{M}{6L}\left(L^2 - 3b^2\right)$ $EI\theta(x) = \frac{M}{2L}x^2 - M\langle x-a\rangle + EI\theta_A$ $EI\delta(x) = \frac{M}{6L}x^3 - \frac{M}{2}\langle x-a\rangle^2 + EI\theta_A x$
	$V_A = \frac{wc}{L}\left(L - b + \frac{c}{2}\right)$ $EI\theta_A = \frac{V_A}{6}L^2 + \frac{w}{24L}\left[(L-b)^4 - (L-a)^4\right]$ $EI\theta(x) = \frac{V_A}{2}x^2 - \frac{w}{6}\langle x-a\rangle^3 + \frac{w}{6}\langle x-b\rangle^3 + EI\theta_A$ $EI\delta(x) = \frac{V_A}{6}x^3 - \frac{w}{24}\langle x-a\rangle^4 + \frac{w}{24}\langle x-b\rangle^4 + EI\theta_A x$